# Nonholonomic Constraints with Fractional Derivatives

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**Abstract**

We consider the fractional generalization of nonholonomic constraints defined by equations with fractional derivatives and provide some examples. The corresponding equations of motion are derived using variational principle.

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# 1 Introduction

The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grunwald, and Letnikov [1, 2, 3]. Fractional analysis proved to be useful in mechanics and physics. In a fairly short time the list of such applications continuously grows. The applications include chaotic dynamics [4, 5], material sciences [6, 7, 8, 9], mechanics of fractal and complex media [10, 11, 12], quantum mechanics [13, 14], physical kinetics [4, 15, 16, 17], plasma physics [18, 19], electromagnetic theory [20, 21, 19], long-range dissipation [22], non-Hamiltonian mechanics [23, 24], long-range interaction [25, 26], anomalous diffusion and transport theory [4, 27, 28, 29].

In this paper, we consider the fractional generalization of nonholonomic constraints such that the constraint equations consist of fractional derivatives, called fractional constraints. The corresponding equations of motion will be derived by the d'Alembert-Lagrange principle and some simple examples are considered.

In Sec 2, we provide a brief review of nonholonomic systems, fix notations and convenient references. In Sec. 3, we consider fractional generalizations of nonholonomic constraints. Some examples are considered. In Sec. 4, we discuss the applicability of the stationary action principle for fractional constraints. Finally, a short conclusion is given in Sec. 5.

# 2 Nonholonomic Constraints with Integer Derivatives

In this section, a brief review of nonholonomic systems is considered to fix notations and provide convenient references [32].

## 2.1 Lagrange Equations for Nonholonomic System

It is known that the d'Alembert-Lagrange principle allows us to derive equations of motion with holonomic and nonholonomic constraints. For N-particle system it has the form of the variation equation

$$\left( \frac{d(m\mathbf{v}_i)}{dt} - \mathbf{F}_i \right) \delta \mathbf{r}_i = 0, \tag{1}$$

where $\mathbf{r}_i$ $(i = 1...N)$ is a radius-vector of $i$th particle, $\mathbf{v}_i = \dot{\mathbf{r}}_i$ is a velocity, and $\mathbf{F}_i$ is a force that acts on $i$th particle, and the sum over repeated index $i$ is from 1 to $N$. To exclude holonomic constraints, the general coordinates $q^k$ $(k = 1, ...n)$ are used. Here, $n = 3N - m$ is a number of degrees of freedom, where $m$ is a number of holonomic constraints. Then $\mathbf{r}_i$ is a function of generalized coordinates and time: $\mathbf{r}_i = \mathbf{r}_i(q, t)$. Using $\delta \mathbf{r}_i = (\partial \mathbf{r}_i / \partial q^k) \delta q^k$, Eq. (1) gives

$$\left( \frac{d(m\mathbf{v}_i)}{dt} \frac{\partial \mathbf{r}_i}{\partial q^k} - \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q^k} \right) \delta q^k = 0, \tag{2}$$

and sum over repeated index $k$ is from 1 to $n$. Then, we define [30] the generalized forces:

$$Q_k = \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q^k} \quad k = 1, ..., n.$$

By usual transformations [30]

$$\frac{d(m\mathbf{v}_i)}{dt} \frac{\partial \mathbf{r}_i}{\partial q^k} = \frac{d}{dt} \left( m\mathbf{v}_i \frac{\partial \mathbf{r}_i}{\partial q^k} \right) - (m\mathbf{v}_i) \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q^k} =$$

$$= \frac{d}{dt} \left( m\mathbf{v}_i \frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} \right) - m\mathbf{v}_i \frac{\partial \mathbf{v}_i}{\partial q^k} =$$

$$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}^k} \left( \frac{m}{2} \mathbf{v}_i \mathbf{v}_i \right) - \frac{\partial}{\partial q^k} \left( \frac{m}{2} \mathbf{v}_i \mathbf{v}_i \right) = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k},$$

we transform Eq. (2) into

$$\left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k \right) \delta q^k = 0, \tag{3}$$

where $T = m\mathbf{v}^2/2$ is a kinetic energy. Using

$$\mathbf{v}_i = \frac{d\mathbf{r}_i(q, t)}{dt} = \frac{\partial \mathbf{r}_i}{\partial q^k} \frac{dq^k}{dt} + \frac{\partial \mathbf{r}_i}{\partial t}, \tag{4}$$

we get

$$T = \frac{m}{2}\left(g_{kl}(q,t)\dot{q}^k\dot{q}^l + 2g_k(q,t)\dot{q}^k + g(q,t)\right),$$

where

$$g_{kl}(q,t) = \frac{\partial \mathbf{r}_i}{\partial q^k}\frac{\partial \mathbf{r}_i}{\partial q^l}, \quad g_k(q,t) = \frac{\partial \mathbf{r}_i}{\partial q^k}\frac{\partial \mathbf{r}_i}{\partial t}, \quad g(q,t) = \frac{\partial \mathbf{r}_i}{\partial t}\frac{\partial \mathbf{r}_i}{\partial t}. \tag{5}$$

For the nonholonomic constraint,

$$R_k \delta q^k = 0, \tag{6}$$

where $R_k$ is a reaction force of the constraint

$$f(q,\dot{q}) = 0, \tag{7}$$

and the variations $\delta q^k$ are defined [31, 32] by

$$\frac{\partial f}{\partial \dot{q}^k}\delta q^k = 0. \tag{8}$$

Equation (8) is called Chetaev's condition [32]. Comparing Eqs. (6) and (8), we obtain

$$R_k = \lambda \frac{\partial f}{\partial \dot{q}^k}, \tag{9}$$

where $\lambda$ is a Lagrange multiplier. Chetaev's definition of variations states that the actual constrained motion should occur along a trajectory obtained by normal projection of a force onto a constraint hypersurface. The constraint force $R_k$ is minimum when $R_k$ is chosen perpendicular to the constraint surface or parallel to the gradient $\partial f/\partial \dot{q}_k$.

In general, the nonholonomic system is subjected to action of the generalized force $Q_k$ and the constraint force $R_k$. Then the variational equation is

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k - R_k\right)\delta q^k = 0. \tag{10}$$

From (9), we obtain

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} - Q_k - \lambda\frac{\partial f}{\partial \dot{q}^k}\right)\delta q^k = 0. \tag{11}$$

In Eq. (8), we can consider $\delta\dot{q}^s$, $s = 1, 2, ..., n-1$, as independent variations. Then $\delta\dot{q}^n$ is not independent, and Eq. (8) gives

$$\delta q^n = -\left(\frac{\partial f}{\partial \dot{q}^n}\right)^{-1}\sum_{s=1}^{n-1}\frac{\partial f}{\partial \dot{q}^s}\delta q^s.$$

Suppose that $\lambda$ satisfies the equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^n} - \frac{\partial T}{\partial q^n} - Q_n - \lambda\frac{\partial f}{\partial \dot{q}^n} = 0. \tag{12}$$

Then the term with $k = n$ in (11) is equal to zero, and Eq. (11) has $n-1$ terms with $k = 1, ..., n-1$. In Eq. (11), the variations with $k = 1, 2, ..., n-1$ are independent, and the sum is separated on $n-1$ equations. As the result, Eq. (11) is equivalent to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = Q_k + \lambda \frac{\partial f}{\partial \dot{q}^k}, \quad (k = 1, ..., n). \tag{13}$$

Equations (7) and (13) form a system of $n+1$ equations with $n+1$ unknowns $\lambda$ and $q^k$, where $k = 1, ..., n$. Solutions of these equations describe particles motion as a motion of system with nonlinear nonholonomic constraint (7).

## 2.2 Nonholonomic System as a Holonomic One

In this section, we present the equations of motion with nonholonomic constraint as equations for a holonomic system.

The canonical momenta $p^k$ are defined by

$$p_k = \frac{\partial T}{\partial \dot{q}^k} = m g_{kl}(q,t)\dot{q}^l + m g_k(q,t), \quad (k = 1, ..., n). \tag{14}$$

Using (14), we can define

$$\tilde{f}(p,q,t) = f(\dot{q}(q,p,t), q, t). \tag{15}$$

Suppose that the constraint is integral of motion. Then the total time derivative of (15) gives

$$\frac{d\tilde{f}}{dt} = 0, \quad \frac{\partial \tilde{f}}{\partial p_k}\dot{p}_k + \frac{\partial \tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial \tilde{f}}{\partial t} = 0. \tag{16}$$

Substitution of (13) into (16) get

$$\frac{\partial \tilde{f}}{\partial p_k}\left(\frac{\partial T}{\partial q^k} + Q_k + \lambda \frac{\partial \tilde{f}}{\partial \dot{q}^k}\right) + \frac{\partial \tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial \tilde{f}}{\partial t} = 0. \tag{17}$$

From Eq. (17), we obtain

$$\lambda = -\left(\frac{\partial \tilde{f}}{\partial p_k}\frac{\partial \tilde{f}}{\partial \dot{q}^k}\right)^{-1}\left(\frac{\partial \tilde{f}}{\partial p_k}\left(\frac{\partial T}{\partial q^k} + Q_k\right) + \frac{\partial \tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial \tilde{f}}{\partial t}\right). \tag{18}$$

Then the Lagrange equations (13) have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = Q_k - \frac{\partial \tilde{f}}{\partial \dot{q}^k}\left(\frac{\partial \tilde{f}}{\partial p_k}\frac{\partial \tilde{f}}{\partial \dot{q}^k}\right)^{-1}\left(\frac{\partial \tilde{f}}{\partial p_k}\left(\frac{\partial T}{\partial q^k} + Q_k\right) + \frac{\partial \tilde{f}}{\partial q^k}\dot{q}^k + \frac{\partial \tilde{f}}{\partial t}\right). \tag{19}$$

Equations (19) describes the motion of a holonomic system with $n$ degrees of freedom. For any trajectory of the system in the phase space, we have $\tilde{f} = 0$. If the initial values $q_k(0)$ and $\dot{q}_k(0)$

satisfy the constraint condition $f(q(0), \dot{q}(0), t_0) = 0$, then the solution of Eq. (19) is a motion of the nonholonomic system.

Let us define a generalized force $\Lambda_k = Q_k + R_k$, which depends on generalized velocities $\dot{q}^k$, generalized coordinates $q^k$, and time $t$. If

$$\frac{\partial \Lambda_k}{\partial \dot{q}^m} + \frac{\partial \Lambda_m}{\partial \dot{q}^k} = 0,$$

$$\frac{\partial \Lambda_k}{\partial q^m} + \frac{\partial \Lambda_m}{\partial q^k} = \frac{1}{2} \frac{d}{dt} \left( \frac{\partial \Lambda_k}{\partial \dot{q}^m} - \frac{\partial \Lambda_m}{\partial \dot{q}^k} \right),$$

known as the Helmholtz conditions, are satisfied, then a generalized potential $U = U(\dot{q}, q, t)$ exists and

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}^k} - \frac{\partial U}{\partial q^k} = \Lambda_k.$$

In this case, the Hamilton variational principle has the form of the stationary action principle. In order to use this principle for a nonholonomic system, we should consider such trajectories that their initial conditions satisfy the constraint equation (7).

Note that nonholonomic constraint (7) and non-potential generalized force $Q_k$ can be compensated such that resulting generalized force $\Lambda_k$ is a generalized potential force, and system is a Lagrangian and non-dissipative system with holonomic constraints.

# 3 Constraints with Fractional Derivatives

## 3.1 Fractional Derivatives

The fractional derivative has different definitions [1, 2], and exploiting any of them depends on the kind of the problems, initial (boundary) conditions, and the specifics of the considered physical processes. The classical definition is the so-called Riemann-Liouville derivative [1, 2]

$${}_a\mathcal{D}_t^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_a^x \frac{f(z)dz}{(x-z)^{\alpha-m+1}},$$

$${}_t\mathcal{D}_b^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_x^b \frac{f(z)dz}{(z-x)^{\alpha-m+1}}, \tag{20}$$

where $m - 1 < \alpha < m$. Due to reasons, concerning the initial conditions, it is more convenient to use the Caputo fractional derivatives [7, 35, 36]. Its main advantage is that the initial conditions take the same form as for integer-order differential equations.

**Definition.** *The Caputo fractional derivatives are defined by the equations*

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{21}$$

$$ {}_tD^{\alpha}_b f(t) = \frac{1}{\Gamma(m-\alpha)} \int_t^b \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad (22) $$

*where* $m-1 < \alpha < m$, *and* $f^{(m)}(\tau) = d^m f(\tau)/d\tau^m$.

**Proposition 1.** *The total time derivative of the Caputo fractional derivative of order* $\alpha$ *can be presented as a fractional derivative of order* $\alpha+1$ *by*

$$ \frac{d}{dt}\, {}_aD^{\alpha}_t f(t) =_a D^{\alpha+1}_t f(t) + \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} f^{(m)}(a), \quad (23) $$

*where* $m = [\alpha]+1$, *and* $[...]$ *means floor function.*

**Proof**. The definition (21) can be presented in the form

$$ {}_aD^{\alpha}_t f = \ {}_aJ^{m-\alpha}_t D^m_t f, \quad (24) $$

where ${}_aJ^{m-\alpha}_t$ is a fractional integral

$$ {}_aJ^{\varepsilon}_t f(t) = \frac{1}{\Gamma(\varepsilon)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\varepsilon}} d\tau. \quad (25) $$

The operations $D^1_t$ and $J^{\varepsilon}_t$ do not commute:

$$ D^1_t\ {}_aJ^{\varepsilon}_t f(t) =_a J^{\varepsilon}_t D^1_t f(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f(a). \quad (26) $$

From (24), we have

$$ \frac{d}{dt}\, {}_aD^{\alpha}_t f(t) = D^1_t\ {}_aJ^{\varepsilon}_t D^m_t f(t) = D^1_t\ {}_aJ^{\varepsilon}_t f^{(m)}(t), \quad (27) $$

where ${}_aJ^{\varepsilon}_t$ is a fractional integration of order $\varepsilon = m-\alpha$. Using (26) and (27), we get

$$ D^1_t J^{\varepsilon}_t D^m_t f(t) =_a J^{\varepsilon}_t D^1_t f^{(m)}(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f^{(m)}(a) = $$

$$ = \ {}_aJ^{\varepsilon}_t f^{(m+1)}(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f^{(m)}(a) = \ {}_aD^{\alpha+1}_t f(t) + \frac{t^{\varepsilon-1}}{\Gamma(\varepsilon)} f^{(m)}(a). \quad (28) $$

Substitution of (28) into (27) proves (23).

## 3.2 Fractional Equations of Motion

Asume that the constraint equation has fractional derivatives:

$$f(q, \dot{q}, \ {}_aD_t^\alpha q, \ {}_tD_b^\alpha q) = 0, \tag{29}$$

i.e. it is a fractional differential equation [3]. Such constraint can be called fractional nonholonomic constraint. Since Eq. (29) has also derivatives of integer order, we can use the Chetaev definition of variation (8) and the Lagrange equations (13). For generalized potential forces

$$Q_k = \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_k} - \frac{\partial U}{\partial q_k}, \quad (k = 1, ..., n),$$

and we can rewrite Eq. (13) as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \lambda \frac{\partial f}{\partial \dot{q}_k}, \quad (k = 1, ..., n), \tag{30}$$

where $L = T(q, \dot{q}) - U(q, \dot{q})$ is the Lagrangian. To simplify our calculations, we consider

$$L = L(q, \dot{q}) = \frac{1}{2}\sum_{k=1}^{n} (\dot{q}_k)^2 - u(q), \tag{31}$$

where $u(q)$ is a potential energy of the system. Then, Eq. (30) becomes

$$\ddot{q}_k = -\frac{\partial u}{\partial q_k} + \lambda \frac{\partial f}{\partial \dot{q}_k}, \quad (k = 1, ..., n). \tag{32}$$

Suppose that the constraint is an integral of motion, i.e., $df/dt = 0$. Then

$$\frac{\partial f}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial f}{\partial ({}_aD_t^\alpha q_k)}\frac{d({}_aD_t^\alpha q_k)}{dt} + \frac{\partial f}{\partial ({}_tD_b^\alpha q_k)}\frac{d({}_tD_b^\alpha q_k)}{dt} + \frac{\partial f}{\partial q_k}\frac{dq_k}{dt} = 0. \tag{33}$$

Equation (33) can be presented as

$$\frac{\partial f}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial f}{\partial ({}_aD_t^\alpha q_k)} D_t^1 \ {}_aD_t^\alpha q_k + \frac{\partial f}{\partial ({}_tD_b^\alpha q_k)} D_t^1 \ {}_tD_b^\alpha q_k + \frac{\partial f}{\partial q_k}\dot{q}_k = 0. \tag{34}$$

Substitution of (32) into (34) gives

$$\frac{\partial f}{\partial \dot{q}_k}\left(-\frac{\partial u}{\partial q_k} + \lambda \frac{\partial f}{\partial \dot{q}_k}\right) + \frac{\partial f}{\partial ({}_aD_t^\alpha q_k)} D_t^1 \ {}_aD_t^\alpha q_k + \frac{\partial f}{\partial ({}_tD_b^\alpha q_k)} D_t^1 \ {}_tD_b^\alpha q_k + \frac{\partial f}{\partial q_k}\dot{q}_k = 0. \tag{35}$$

From this equation, one can obtain the Lagrange multiplier $\lambda$:

$$\lambda = \left(\frac{\partial f}{\partial \dot{q_m}}\frac{\partial f}{\partial \dot{q}_m}\right)^{-1}\left(\frac{\partial f}{\partial \dot{q}_l}\frac{\partial u}{\partial q_l} - \frac{\partial f}{\partial ({}_aD_t^\alpha q_l)} D_t^1 \ {}_aD_t^\alpha q_l - \frac{\partial f}{\partial ({}_tD_b^\alpha q_l)} D_t^1 \ {}_tD_b^\alpha q_l - \frac{\partial f}{\partial q_l}\dot{q}_l\right). \tag{36}$$

Insertion of Eq. (36) into Eq. (32) yields

$$\ddot{q}_k = -\frac{\partial u}{\partial q_k} + \frac{\partial f}{\partial \dot{q}_k}\left(\frac{\partial f}{\partial \dot{q}_m}\frac{\partial f}{\partial \dot{q}_m}\right)^{-1}\left(\frac{\partial f}{\partial \dot{q}_l}\frac{\partial u}{\partial q_l} - \frac{\partial f}{\partial({}_aD^{\alpha}_t q_l)}D^1_t\ {}_aD^{\alpha}_t q_l - \frac{\partial f}{\partial({}_tD^{\alpha}_b q_l)}D^1_t\ {}_tD^{\alpha}_b q_l - \frac{\partial f}{\partial q_l}\dot{q}_l\right). \tag{37}$$

These equations describe holonomic system that is equivalent to the nonholonomic one with fractional constraint. For any motion of the system, we have $f = 0$. If the initial values satisfy the constraint condition $f(q(0), \dot{q}(0),\ {}_aD^{\alpha}_t q(0),\ {}_tD^{\alpha}_b q(0)) = 0$, then the solution of Eq. (37) describes a motion of the system (31) with fractional constraint (29).

### 3.3 Linear Fractional Constraint

Suppose that the constraint (29) is linear with respect to integer derivatives $\dot{q}_k$, i.e.,

$$f = a_k\dot{q}_k + \beta(\ {}_aD^{\alpha}_t q,\ {}_tD^{\alpha}_b q, q). \tag{38}$$

In this case, $R_k = a_k$, and Eqs. (37) can be presented as

$$\ddot{q}_k = -\sum_{l=1}^{n}\left(\delta_{kl} - \frac{a_k a_l}{a^2}\right)\frac{\partial u}{\partial q_l} - \frac{a_k}{a^2}\sum_{l=1}^{n}\left(\frac{\partial f}{\partial({}_aD^{\alpha}_t q_l)}D^1_t\ {}_aD^{\alpha}_t q_l + \frac{\partial f}{\partial({}_tD^{\alpha}_b q_l)}D^1_t\ {}_tD^{\alpha}_b q_l + \frac{\partial \beta}{\partial q_l}\dot{q}_l\right), \tag{39}$$

where $a^2 = \sum_{k=1}^{n} a_k a_k$. If

$$\beta({}_aD^{\alpha}_t q,\ {}_tD^{\alpha}_b q, q) = b_k\ {}_aD^{\alpha}_t q_k, \tag{40}$$

then

$$f = a_k\dot{q}_k + b_k\ {}_aD^{\alpha}_t q_k. \tag{41}$$

This constraint is linear with respect to integer derivative $\dot{q}_k$ and fractional derivatives ${}_aD^{\alpha}_t q_k$. Then the equations of motion are

$$\ddot{q}_k = -\sum_{l=1}^{n}\left(\delta_{kl} - \frac{a_k a_l}{a^2}\right)\frac{\partial u}{\partial q_l} - \sum_{l=1}^{n}\frac{a_k b_l}{a^2}D^1_t\ {}_aD^{\alpha}_t q_l. \tag{42}$$

Using Proposition 1, we obtain

$$\ddot{q}_k = -\sum_{l=1}^{n}\left(\delta_{kl} - \frac{a_k a_l}{a^2}\right)\frac{\partial u}{\partial q_l} - \sum_{l=1}^{n}\frac{a_k b_l}{a^2}\left({}_aD^{\alpha+1}_t q_l + \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)}q_l^{(m)}(a)\right). \tag{43}$$

where $q^{(m)}(a) = (D^m_t q(t))_{t=a}$. As the result, we get the fractional equations of motion with Caputo derivative of order $\alpha+1$. The nonholonomic systems (42) with integer $\alpha$ are considered in Refs. [59, 60].

### 3.4 One-dimensional Example

In one-dimensional case ($n = 1$), Eq. (42) with ${}_0D_t^{\alpha} q$ has the form

$$\ddot{q} = -\frac{b_1}{a_1} D_t^1 \, {}_0D_t^{\alpha} q. \tag{44}$$

Eq. (44) can be presented as

$$D_t^1[\dot{q} + (b_1/a_1)_0 D_t^{\alpha} q] = 0. \tag{45}$$

Then

$$\dot{q} + (b_1/a_1)_0 D_t^{\alpha} q = C_0. \tag{46}$$

Supposing $\alpha > 1$, and using proposition 1, we get

$$D_t^1[q + (b_1/a_1)_0 D_t^{\alpha-1} q] = \frac{b_1 t^{m-\alpha}}{a_1 \Gamma(m-\alpha+1)} q(0) + C_0. \tag{47}$$

As the result, we obtain

$${}_0D_t^{\alpha-1} q + (a_1/b_1) q = \frac{t^{m-\alpha+1}}{\Gamma(m-\alpha+2)} q(0) + C_1 t + C_2, \tag{48}$$

where we use $x\Gamma(x) = \Gamma(x+1)$, and $C_1 = C_0 a_1/b_1$.

For $2 < \alpha < 3$, Eq. (48) describes the linear fractional oscillator

$${}_0D_t^{\alpha-1} q(t) + \omega^2 q(t) = Q(t), \tag{49}$$

where $\omega^2 = (a_1/b_1)$ is dimensionless "frequency", and $Q(t)$ is the external force:

$$Q(t) = \frac{t^{m-\alpha+1}}{\Gamma(m-\alpha+2)} q(0) + C_1 t + C_2.$$

The Caputo fractional derivative ${}_0D_t^{\alpha-1}$ allows us to use the regular initial conditions [3] for Eq. (49). The linear fractional oscillator is an object of numerous investigations [37, 38, 39, 40, 41, 42, 43, 44, 45, 46] because of different applications.

The exact solution [37, 38] of Eq. (49) for $2 < \alpha < 3$ is

$$q(t) = q(0) E_{\alpha-1,1}(-\omega^2 t^{\alpha-1}) + t q'(0) E_{\alpha-1,2}(-\omega^2 t^{\alpha-1}) - \int_0^t Q(t-\tau) \dot{q}_0(\tau) d\tau, \tag{50}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{51}$$

is the generalized two-parameter Mittag-Leffler function [47, 48], and

$$q_0(\tau) = E_{\alpha-1,1}(-\omega^2 \tau^{\alpha-1}).$$

The decomposition of (50) is [37]:

$$q(t) = q(0)\left[f_{\alpha,0}(t) + g_{\alpha,0}(t)\right] + t\dot{q}(0)\left[f_{\alpha,1}(t) + g_{\alpha,1}(t)\right] - \int_0^t Q(t-\tau)\dot{q}_0(\tau)d\tau, \tag{52}$$

where

$$f_{\alpha,k}(t) = \frac{(-1)^k}{\pi}\int_0^\infty e^{-rt}\frac{r^{\alpha-1-k}\sin(\pi\alpha)}{r^{2\alpha}+2r^\alpha\cos(\pi\alpha)+1}dr,$$

$$g_{\alpha,k}(t) = \frac{2}{\alpha}e^{t\cos(\pi/\alpha)}\ \cos\left[t\sin(\pi/\alpha) - \pi k/\alpha\right], \quad (k=0,1). \tag{53}$$

For the initial conditions $q(0)=1$, and $\dot{q}(0)=0$:

$$q(t) = E_\alpha(-t^\alpha) = f_{\alpha,0}(t) + g_{\alpha,0}(t) - \int_0^t Q(t-\tau)[\dot{f}_{\alpha,0}(\tau) + \dot{g}_{\alpha,0}(\tau)]d\tau. \tag{54}$$

The first term in (54) decay in power law with time while the second term decays exponentially [37, 38, 40, 41].

## 3.5 Two-dimensional Examples

In the two-dimensional case ($n=2$), Eq. (42) has the form

$$\ddot{q}_1 = -\frac{a_2^2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_1} + \frac{a_1a_2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_2} - \frac{a_1b_1}{a_1^2+a_2^2}D_t^1\ {}_aD_t^\alpha q_1 - \frac{a_1b_2}{a_1^2+a_2^2}D_t^1\ {}_aD_t^\alpha q_2; \tag{55}$$

$$\ddot{q}_2 = -\frac{a_1^2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_2} + \frac{a_1a_2}{a_1^2+a_2^2}\frac{\partial u}{\partial q_1} - \frac{a_2b_1}{a_1^2+a_2^2}D_t^1\ {}_aD_t^\alpha q_1 - \frac{a_2b_2}{a_1^2+a_2^2}D_t^1\ {}_aD_t^\alpha q_2. \tag{56}$$

Let us consider the special cases of these equations.

(1) Suppose $a_1 = 0$; then

$$\ddot{q}_1 = -\frac{\partial u}{\partial q_1}; \quad \ddot{q}_2 = -\frac{b_1}{a_2}D_t^1\ {}_aD_t^\alpha q_1 - \frac{b_2}{a_2}D_t^1\ {}_aD_t^\alpha q_2. \tag{57}$$

If $a_1 = 0$, and $b_2 = 0$, then (57) are

$$\ddot{q}_1 = -\frac{\partial u}{\partial q_1}, \quad \ddot{q}_2 = -\frac{b_1}{a_2}D_t^1\ {}_aD_t^\alpha q_1. \tag{58}$$

(2) Suppose $b_1 = 0$ and $a_1 = a_2 = c$; then we have

$$\ddot{q}_1 = -\frac{1}{2}\frac{\partial u}{\partial q_1} + \frac{1}{2}\frac{\partial u}{\partial q_2} - \frac{b_2}{2c}D_t^1\ {}_aD_t^\alpha q_2; \tag{59}$$

$$\ddot{q}_2 = -\frac{1}{2}\frac{\partial u}{\partial q_2} + \frac{1}{2}\frac{\partial u}{\partial q_1} - \frac{b_2}{2c}D_t^1\ {}_aD_t^\alpha q_2. \tag{60}$$

Using

$$x = \frac{q_1 + q_2}{2}, \quad y = \frac{q_1 - q_2}{2}, \quad g = b_2/c,$$

we can rewrite Eqs. (59) and (60) in the form

$$\ddot{x} = -gD_t^1 \ {}_aD_t^\alpha x + gD_t^1 \ {}_aD_t^\alpha y, \quad \ddot{y} = -\frac{\partial U}{\partial y}, \tag{61}$$

where $U(x, y) = u(q_1, q_2) = u(x + y, x - y)$. If $U = K(x)y + s(x)$, then Eq. (61) is

$$\ddot{x} = -gD_t^1 \ {}_aD_t^\alpha x + gD_t^1 \ {}_aD_t^\alpha y, \quad \ddot{y} = -K(x). \tag{62}$$

Using $D_t^\alpha y = D_t^{\alpha-2}\ddot{y}$, Eq. (62) gives

$$\ddot{x} = -gD_t^1 \ {}_aD_t^\alpha x - gD_t^1 \ {}_aD_t^{\alpha-2}K(x). \tag{63}$$

Then

$$D_t^1[\dot{x} + g \ {}_aD_t^\alpha x + g \ {}_aD_t^{\alpha-2}K(x)] = 0.$$

For $\alpha > 2$, then we can get

$$\dot{x} + g_a^{-1}D_t^\alpha x + g_a D_t^{\alpha-2}K(x) + C = 0, \tag{64}$$

where $C$ is a constant. Using ${}_aD_t^\varepsilon \ {}_aJ_t^\varepsilon f(t) = f(t)$, and ${}_aD_t^\varepsilon \ {}_aD_t^\alpha f(t) = D_t^m f(t)$, where $\varepsilon = m - \alpha$, Eq. (64) can be written as

$$D^\varepsilon x + g^{-1}x^{(m)} + g_a D_t^{m-2}K(x) = 0. \tag{65}$$

If $m = 2$ $(1 < \alpha < 2)$, then

$$g^{-1}\ddot{x} + D^\varepsilon x + g_a K(x) = 0. \tag{66}$$

This equation can be considered as an equation of nonlinear fractional oscillator [40, 41], where fractional derivative describes the power dumping.

# 4 Fractional Conditional Extremum

## 4.1 Extremum for Fractional Constraint

Let us consider the stationary value of an action integral

$$\delta \int_a^b dt \ L(q, \dot{q}) = 0,$$

for the lines that satisfy the constraint equation $f(q, \dot{q}) = 0$. Using the Lagrange multiplier $\mu = \mu(t)$, we get a variational equation

$$\delta \int_a^b dt \, [L(q, \dot{q}) + \mu f(q, \dot{q})] = 0.$$

Then the Euler-Lagrange equations [34] are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \mu \left( \frac{\partial f}{\partial q_k} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}_k} \right) - \dot{\mu}\frac{\partial f}{\partial \dot{q}_k}, \quad (k = 1, ..., n). \tag{67}$$

Note that these equations consist of the derivative of Lagrange multiplier $\dot{\mu}$. The proof of Eq. (67) is realized in [34].

For the fractional constraint

$$f(q, \dot{q},\ {}_aD_t^\alpha q,\ {}_tD_b^\alpha q) = 0, \tag{68}$$

we can define the Lagrangian as

$$\mathcal{L}(q, \dot{q},\ {}_aD_t^\alpha q,\ {}_tD_b^\alpha q, \lambda) = L(q, \dot{q}) + \mu(t) f(q, \dot{q},\ {}_aD_t^\alpha q,\ {}_tD_b^\alpha q). \tag{69}$$

Using the Agrawal variational equation [33], we obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} +_a D_t^\alpha \frac{\partial \mathcal{L}}{\partial ({}_aD_t^\alpha q_k)} +_t D_b^\alpha \frac{\partial \mathcal{L}}{\partial ({}_tD_b^\alpha q_k)} = 0, \quad (k = 1, ..., n). \tag{70}$$

Substitution of Eq. (69) into Eq. (70) gives

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \mu \frac{\partial f}{\partial q_k} + \ {}_aD_t^\alpha \left( \mu \frac{\partial f}{\partial ({}_aD_t^\alpha q_k)} \right) + \ {}_tD_b^\alpha \left( \mu \frac{\partial f}{\partial ({}_tD_b^\alpha q_k)} \right) - \frac{d}{dt}\left( \mu \frac{\partial f}{\partial \dot{q}_k} \right) = 0. \tag{71}$$

These equations describe the fractional conditional extremum.

Let us consider applicability of stationary action principle for mechanical systems with fractional nonholonomic constraints. The equations of motion are derived from the d'Alembert-Lagrange principle. The fractional conditional extremum can be obtained from stationary action principle. In general, these equations are not equivalent [34]. The condition of this equivalence for fractional constraints is suggested in the proposition.

**Proposition 2.** *Equations (30) and (71) for nonholonomic system with fractional constraint (68) have the equivalent set of solutions if the conditions*

$$\left[ \mu \frac{\partial f}{\partial q_k} + \ {}_aD_t^\alpha \left( \mu \frac{\partial f}{\partial_a D_t^\alpha q_k} \right) + \ {}_tD_b^\alpha \left( \mu \frac{\partial f}{\partial_t D_b^\alpha q_k} \right) - \frac{d}{dt}\left( \mu \frac{\partial f}{\partial \dot{q}_k} \right) \right] \delta q_k = 0, \quad \frac{\partial f}{\partial \dot{q}_k} \delta q_k = 0. \tag{72}$$

*are satisfied.*

**Proof**. To prove the proposition, we multiply Eqs. (30) and (71) on the variation $\delta q^k$ and consider a sum with respect to $k$:

$$\left( \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \right) \delta q_k = \lambda \frac{\partial f}{\partial \dot{q}_k} \delta q_k, \tag{73}$$

$$\left(\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k+\left[\mu\frac{\partial f}{\partial q_k}+\ {}_aD^\alpha_t\left(\mu\frac{\partial f}{\partial({}_aD^\alpha_t q_k)}\right)+\ {}_tD^\alpha_b\left(\mu\frac{\partial f}{\partial({}_tD^\alpha_b q_k)}\right)-\frac{d}{dt}\left(\mu\frac{\partial f}{\partial \dot{q}_k}\right)\right]\delta q_k=0, \tag{74}$$

From the definition of variations (8), Eq. (73) is

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}\right)\delta q_k=0, \tag{75}$$

Substituting Eq. (75) into Eq. (74), we obtain (72).

It is known [34], that stationary action principle cannot be derived from the d'Alembert-Lagrange principle for wide class of nonholonomic and non-Hamiltonian systems. The same can be applied to the case of nonlinear fractional nonholonomic constraints.

## 4.2 Hamilton's Approach

Using Eq. (70), we define the momenta

$$p_k=\frac{\partial \mathcal{L}}{\partial \dot{q}_k}=\frac{\partial L}{\partial \dot{q}_k}+\mu\frac{\partial f}{\partial \dot{q}_k}, \tag{76}$$

and the Hamiltonian

$$\mathcal{H}(q,p)=p_k\dot{q}-\mathcal{L}, \tag{77}$$

where $\mathcal{L}=L+\mu f$. Eq. (70) gives

$$\frac{dp_k}{dt}=\frac{\partial \mathcal{H}}{\partial q_k}+\ {}_aD^\alpha_t\left(\mu\frac{\partial f}{\partial({}_aD^\alpha_t q_k)}\right)+\ {}_tD^\alpha_b\left(\mu\frac{\partial f}{\partial({}_tD^\alpha_b q_k)}\right). \tag{78}$$

To simplify our calculations, we consider the Lagrangian

$$L=\frac{1}{2}(\dot{q})^2-u(q),$$

and the fractional nonholonomic constraint

$$f=A_k(q,\ {}_aD^\alpha_t q)\ \dot{q}_k=0, \quad (\alpha\neq 1). \tag{79}$$

From Eqs. (76) and (79), we obtain

$$p_k=\dot{q}_k+\mu A_k(q,\ {}_aD^\alpha_t q). \tag{80}$$

Then the Hamilton equations are

$$\dot{q}_k=p_k-\mu A_k(q,\ {}_aD^\alpha_t q), \tag{81}$$

$$\dot{p}_k = -\frac{\partial u(q)}{\partial q_k} + \mu(t)\dot{q}_l \frac{\partial A_l}{\partial q_k} + {}_aD_t^\alpha \left( \mu \dot{q}_l \frac{\partial A_l({}_aD_t^\alpha q)}{\partial_a D_t^\alpha q_k} \right). \tag{82}$$

To find the Lagrange multiplier $\mu = \mu(t)$, we multiply Eq. (80) on the functions $a_k$ and consider the sum with respect to $k$:

$$A_k p_k = A_k \dot{q}_k + \mu A_k A_k = \mu A^2. \tag{83}$$

Here, we use the constraint (79), and the notation $A^2 = A_k A_k$, where $A_k = A_k(q, {}_aD_t^\alpha q)$. From (83), we get

$$\mu = \frac{A_k p_k}{A^2}. \tag{84}$$

Substitution of (84) into Eqs. (81) and (82) gives

$$\dot{q}_k = \left( \delta_{kl} - \frac{A_k A_l}{A^2} \right) p_l, \tag{85}$$

$$\dot{p}_k = -\frac{\partial u(q)}{\partial q_k} + \frac{A_m p_m}{A^2} \frac{\partial A_l}{\partial q_k} \dot{q}_l + {}_aD_t^\alpha \left( \frac{A_m p_m}{A^2} \frac{\partial A_l}{\partial_a D_t^\alpha q_k} \dot{q}_l \right). \tag{86}$$

If $a_k = 0$, then we have usual equations of motion for Hamiltonian systems. Note that we derive Hamiltonian equations from Euler-Lagrange equations without using the Legendre transformation, which is typically used.

# 5 Conclusion

The classical mechanics of nonholonomic systems has recently been employed to study a wide variety of problems in the molecular dynamics [49]. In molecular dynamics calculations, nonholonomic systems can be exploited to generate statistical ensembles as the canonical, isothermal-isobaric and isokinetic ensembles [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60]. Using fractional nonholonomic constraints, we can consider a fractional extension of the statistical mechanics of conservative Hamiltonian systems to a much broader class of systems. Let us point out some nonholonomic systems that can be generalized by using the nonholonomic constraint with fractional derivatives.

(1) In the papers [50, 51, 52, 53], the constant temperature systems with minimal Gaussian constraint are considered. These systems are non-Hamiltonian ones and they are described by the non-potential forces that are proportional to the velocity, and the Gaussian nonholonomic constraint. Note that this constraint can be represented as an addition term to the non-potential force [60].

(2) In the papers [59, 60], the canonical distribution is considered as a stationary solution of the Liouville equation for a wide class of non-Hamiltonian system. This class is defined by a very simple condition: the power of the non-potential forces must be proportional to the velocity of the Gibbs phase (elementary phase volume) change. This condition defines the general constant temperature systems. Note that the condition is a nonholonomic constraint.

This constraint leads to the canonical distribution as a stationary solution of the Liouville equations. For the linear friction, we derived the constant temperature systems. A general form of the non-potential forces is derived in Ref. [60].


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